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An exact solution to steady heat conduction in a two-dimensional slab on a one-dimensional fin: application to frosted heat exchangers

Y. Xia, A.M. Jacobi *

Department of Mechanical and Industrial Engineering, University of Illinois at Urbana-Champaign, 1206 West Green Street, Urbana, IL 61801, USA

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Abstract

The thermal conditions for frost forming on a metallic fin are considered, and a numerical solution to the two-dimensional conduction problem is used to identify the parametric space for which the problem reduces to a two-dimensional slab on a one-dimensional fin. An analytical solution gives rise to an eigenvalue problem that requires an unusual scalar product definition. A one-term approximation to the new analytical solution provides fin efficiency calculations useful for a range of conditions, including most frosted-coated metallic fins. The series solution and the one-term approximation are of general applicability to low-conductivity coatings on high-conductivity fins.

1. Introduction

This work is motivated by a desire to have a convenient expression for fin efficiency, spanning operating conditions which cause frost to form on the air-side surface of a heat exchanger. In particular, we desire an expression applicable to frosted, flat-tube, heat exchangers with constant-area fins. Such heat exchangers are used for air-cooling applications, and when air flows through a heat exchanger and the fin surface temperature is less than 0 °C, frost can form on the surface. Using an effective conductivity for the frost, the frost and fin material can be considered as a composite medium. Although our study was directed at the problem of frost on a metallic fin, and example calculations are developed for that case, the mathematical analysis is presented in a general fashion and is applicable to a wide range of related problems.

Exact solutions for heat conduction in composite slabs have been provided by several authors. Tittle [1] formulated a one-dimensional orthogonal expansion for composite media, and Padovan [2] developed a general procedure for solving Sturm–Liouville problems arising from transient heat conduction in composite and anisotropic domains. Using a Green's function approach, Huang and Chang [3] provided exact solutions for unsteady, periodic, and steady conduction in composites of infinite, semi-infinite, and finite laminates. Feijoo et al. [4] analytically solved for temperature distributions in a composite fin for a slab with a symmetric internal heat source. Yan et al. [5] obtained series solutions for three-dimensional temperature distributions in two-layer composites, for a range of boundary conditions. Recently, Aviles-Ramos et al. [6] provided an exact solution to the temperature distribution in a two-layer body: one orthotropic and the other isotropic. In most cases, the two- or three-dimensional solution converges slowly, and the computation process can be difficult.

More closely related to the current work, the analytical solution for heat conduction in a composite fin under the usual conditions of constant heat transfer coefficient and uniform ambient temperature has also

* Corresponding author. Tel.: +1-217-333-4108; fax: +1-217-244-6534.
E-mail addresses: xia@students.uiuc.edu (Y. Xia), a-jacobi@uiuc.edu (A.M. Jacobi).

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Nomenclature

Bi	Biot number, as defined by Eq. (1)
h	convective heat transfer coefficient ($W\,m^{-2}\,°C^{-1}$)
k	thermal conductivity ($W\,m^{-1}\,°C^{-1}$)
L	fin length; half the distance between tubes, see Fig. 1 (m)
M	dimensionless group defined in Eq. (17)
R	ratio of thermal resistance defined by Eq. (2)
t	half fin thickness, see Fig. 1 (m)
T	temperature (°C)

Greek symbols

δ	frost thickness (m)
ε	a small parameter, see Eq. (10)
η	fin efficiency, see Eq. (27)
η'	fin efficiency approximation, from a one-term approximation, see Eq. (35)

Subscripts and superscripts

1	in the material of the fin
2	in the material of the frost
b	at the fin base
e	in the environment
B	per Barker [7]

been studied. Barker [7] considered a two-layer composite, and obtained an analytical solution for the two-dimensional temperature distribution within the fin and the coating material. Chu et al. [8] used the Laplace transform and eigenfunction expansions to analyze transient conduction in a composite fin. The resulting expressions were complex, and the inverse transform was difficult to find. The complexity in these two solutions—that of Barker and that of Chu et al.—would be much reduced if one-dimensional heat conduction prevails within one of the two slabs. In a recent study, Mokheimer et al. [9] considered a one-dimensional slab on a two-dimensional fin and obtained numerical and analytical solutions for heat conduction in the composite. They did not explore the limitations of their analysis or assess the potential effects of transverse temperature gradients in a two-dimensional slab.

An exact solution for a two-dimensional slab on a one-dimensional fin has not appeared in the open literature, nor has the applicability of such an approximation been explored for cases such as the one of interest. For most heat exchangers operating under frosting conditions, the fin is thinner than the frost, and both the fin and frost thicknesses are much smaller than the fin length. Furthermore, the frost thermal conductivity is much smaller than the fin conductivity (less than 1% for an aluminum fin). Thus, it is expected that in some cases it will be appropriate to simplify the problem to a two-dimensional slab on a one-dimensional fin; moreover, such an approach is anticipated to yield simpler expressions for temperature and fin efficiency than the case of a two-dimensional slab on a two-dimensional substrate.

In this paper, a numerical solution to conduction within the composite medium comprised of a fin and coating material is used to conduct a parametric study of the effects of geometry, thermal conductivity of the fin and coating material, and convection coefficient on the temperature profiles. In particular, we explore the applicability of assuming a one-dimensional fin with a two-dimensional coating. Next, the exact solution under the assumption of one-dimensional heat flow in the fin and two-dimensional heat flow within the coating is obtained by the method of separation of variables—we obtain an unusual eigenvalue problem, for which a new scalar product is defined for orthogonality. The new solution converges rapidly and its eigenvalues are easily calculated. The exact solution is useful in gaining physical insights into the problem, and it is simple, accurate, and less costly to use than numerical solutions. Furthermore, it will be shown that for many cases, such as for frost on a metallic fin, a simple one-term approximation is valid. Using the one-term approximation, a simple expression for fin efficiency is developed, and this expression is much easier to use than a computational model of the system.

2. Problem description

The physical situation of interest, frost on a metallic fin is shown in Fig. 1(a), where a frosted, flat-tube heat exchanger with constant-area fins is shown in the schematic. The fin depth in the z-direction is large in comparison to lengths in the x- and y-directions. Because temperatures, geometry, and properties in the z-direction are constant, a two-dimensional analysis is used. The dashed box, enlarged in Fig. 1(b), shows in more detail the physical system to be analyzed. The metallic fin and the frost slab form a composite medium. The convection coefficient, free-stream temperature, base temperature, and thermophysical properties are considered as constant.

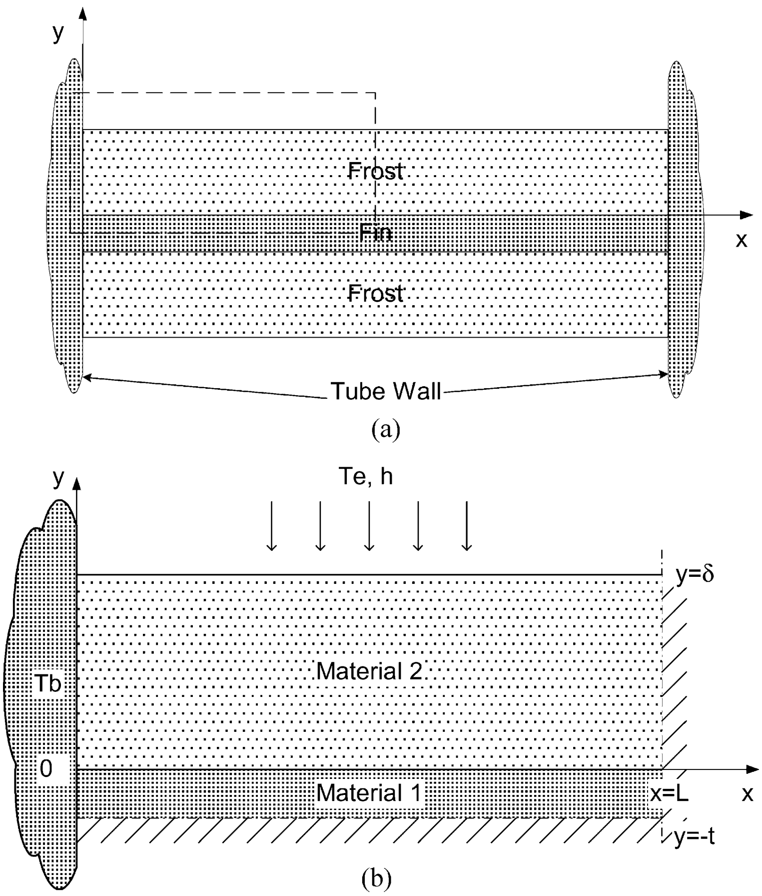

Fig. 1. Schematic of the composite slab, with the one-dimensional fin *material 1*, and the two-dimensional slab *material 2*: (a) the complete physical system, showing a fin between two flat tubes, and (b) simplified system from symmetry.

3. Numerical parametric analysis

The problem of two-dimensional heat flow both for the fin and coating material is solved numerically using well-known finite-difference methods. Consider a Biot number defined as the ratio of internal conduction resistance to the external convection resistance, namely

$$Bi = \frac{h\delta}{k_2}(1+R), \tag{1}$$

where R is the ratio of the transverse thermal resistance of the fin material (t/k_1A, where A is an arbitrary area) to that of the coating material (δ/k_2A), i.e.

$$R = \frac{t/k_1}{\delta/k_2}. \tag{2}$$

Although somewhat arbitrary, we elected to assume one-dimensional conduction if the maximum temperature difference between the outer and inner surfaces was less than 5% of the maximum temperature difference, i.e.,

$$\max_{0\leqslant x\leqslant L}|T_1(x,0)-T_1(x,-t)| < 5\%|T_e - T_b|$$

or

$$\max_{0\leqslant x\leqslant L}|T_2(x,0)-T_2(x,\delta)| < 5\%|T_e - T_b|. \tag{3}$$

We fixed the geometries of the composite slab (L, t and δ), the base and ambient temperature (T_b and T_e), and the convective heat transfer coefficient h, to study the temperature profiles within the two slabs under different combinations of k_1 and k_2 (or Bi and R). The results of this wide-ranging numerical study are provided in Table 1: it gives the parametric range for a valid one-dimensional approximation in either or both slabs. The decision on validity is based on Eq. (3). As an example case, consider $Bi = 10$, and $t = \delta = 0.1L$, then the table shows that an approximation of a one-dimensional fin with a two-dimensional coating is appropriate when $0 < R < 0.2$, or when $32 < R < 63$.

From the results provided in Table 1, we draw the following conclusions: when $Bi < 0.05$, heat conduction within the fin and the coating can both be approximated

Table 1
Temperature profile behavior, under the criteria of Eq. (3), for different combinations of Bi and R

Bi	Geometry t, δ, L	Required R for the modeling approach designated			
		1-D fin; 1-D coating	1-D fin; 2-D coating	2-D fin; 1-D coating	2-D fin; 2-D coating
<0.05	$t = \delta = 0.1L$	$(0, \infty)$	–	–	–
0.1	$t = \delta = 0.1L$	$(\sim 0.35, \infty)$	$(0, \sim 0.35)$	–	–
1.0	$t = \delta = 0.1L$	$(\sim 6.5, \infty)$	$(0, \sim 6.5)$	–	–
1.45	$t = \delta = 0.1L$	$(\sim 9.0, \infty)$	$(0, \sim 1.15) \cup (\sim 1.85, \sim 9.0)$	–	$(\sim 1.15, \sim 1.85)$
10	$t = \delta = 0.1L$	$(\sim 63, \infty)$	$(0, \sim 0.25) \cup (\sim 32, \sim 63)$	–	$(\sim 0.25, \sim 32)$
∞	$t = \delta = 0.1L$	$(\sim 6.3Bi, \infty)$	$(0, \sim 0.25) \cup (\sim 3.2Bi, \sim 6.3Bi)$		$(\sim 0.25, \sim 3.2Bi)$
<~0.05	$t = 0.1\delta = 0.01L$	$(0, \infty)$	–	–	–
0.1	$t = 0.1\delta = 0.01L$	$(\sim 0.01, \infty)$	$(0, \sim 0.01)$	–	–
1.0	$t = 0.1\delta = 0.01L$	$(\sim 5.4, \infty)$	$(0, \sim 5.4)$	–	–
10	$t = 0.1\delta = 0.01L$	$(\sim 63, \infty)$	$(0, \sim 63)$	–	–
100	$t = 0.1\delta = 0.01L$	$(\sim 630, \infty)$	$(0, \sim 630)$	–	–
∞	$t = 0.1\delta = 0.01L$	$(\sim 6.3Bi, \infty)$	$(0, \sim 6.3Bi)$	–	–
<~0.05	$\delta = 0.1t = 0.01L$	$(0, \infty)$	–	–	–
0.1	$\delta = 0.1t = 0.01L$	$(\sim 0.7, \infty)$	$(0, \sim 0.7)$	–	–
1.0	$\delta = 0.1t = 0.01L$	$(\sim 370, \infty)$	$(0, \sim 0.4)$	$(\sim 10, \sim 370)$	$(\sim 0.4, \sim 10)$
10	$\delta = 0.1t = 0.01L$	$(\sim 4400, \infty)$	$(0, \sim 0.18)$	$(\sim 58, \sim 4400)$	$(\sim 0.18, \sim 58)$
100	$\delta = 0.1t = 0.01L$	$(\sim 44,000, \infty)$	$(0, \sim 0.16)$	$(\sim 330, \sim 44,000)$	$(\sim 0.16, \sim 330)$
∞	$\delta = 0.1t = 0.01L$	$(\sim 440Bi, \infty)$	$(0, \sim 0.16)$	$(\sim 3.3Bi, \sim 440Bi)$	$(\sim 0.16, \sim 3.3Bi)$
<~0.05	$\delta = t = 0.01L$	$(0, \infty)$	–	–	–
0.1	$\delta = t = 0.01L$	$(\sim 0.2, \infty)$	$(0, \sim 0.2)$	–	–
1.0	$\delta = t = 0.01L$	$(\sim 3.6, \infty)$	$(0, \sim 3.6)$	–	–
3.5	$\delta = t = 0.01L$	$(\sim 11.0, \infty)$	$(0, \sim 1.4) \cup (\sim 2.8, \sim 11.0)$	–	$(\sim 1.4, \sim 2.8)$
10	$\delta = t = 0.01L$	$(\sim 29, \infty)$	$(0, \sim 0.65) \cup (\sim 15, \sim 29)$	–	$(\sim 0.65, \sim 15)$
∞	$\delta = t = 0.01L$	$(\sim 2.7Bi, \infty)$	$(0, \sim 0.45) \cup (\sim 1.7Bi, \sim 2.7Bi)$	–	$(\sim 0.45, 1.7Bi)$

as one-dimensional in x, for the entire geometrical range studied. For almost all cases in the table, if $Bi > 0.05$, then when $R < 0.1$ the fin can be approximated as one-dimensional, with a two-dimensional coating. This situation represents the parametric range of most importance for frost on a metallic fin. The case of a two-dimensional substrate, with a one-dimensional coating occurs when the fin is thicker than the coating and R is within a certain region (depending on the geometries); however, such cases are not common for the application that motivates this study.

4. Analytical solution for the two-dimensional slab on a one-dimensional fin

On the basis of the parametric study, and in consideration of the motivating problem, the following assumptions are invoked: steady state, two-dimensional conduction in a slab on a one-dimensional fin, with no internal generation, and constant properties. The frost layer is assumed to be of uniform thickness. The base temperature is held at T_b, and there is no contact resistance between the frost and the fin. With these assumptions, the fin temperature T_1 is a function of x only, and the frost temperature is $T_2(x, y)$.

The governing equation for the temperature distribution along the fin, material 1, is

$$k_1 t \frac{d^2 T_1}{dx^2} + k_2 \left.\frac{\partial T_2}{\partial y}\right|_{y=0} = 0 \quad \text{in } 0 < x < L. \tag{4}$$

The diffusion equation in the frost layer, material 2, is

$$\frac{\partial^2 T_2}{\partial x^2} + \frac{\partial^2 T_2}{\partial y^2} = 0 \quad \text{in } 0 < x < L,\ 0 < y < \delta. \tag{5}$$

Eqs. (4) and (5) are subject to the following boundary conditions:

$$\left.\frac{dT_1}{dx}\right|_{x=L} = 0, \quad \left.\frac{\partial T_2}{\partial x}\right|_{x=L} = 0, \tag{6a}$$

$$T_1(0) = T_b, \quad T_2(0, y) = T_b, \tag{6b}$$

$$T_1(x) = T_2(x, 0) \tag{6c}$$

and

$$\left.\frac{\partial T_2}{\partial y}\right|_{y=\delta} = \frac{h}{k_2}(T_e - T_2(x, \delta)). \tag{6d}$$

Combining Eqs. (4)–(6), we obtain the following four boundary conditions for $T_2(x, y)$:

$$\left.\frac{\partial T_2}{\partial x}\right|_{x=L} = 0, \tag{7a}$$

$$T_2(0, y) = T_b, \tag{7b}$$

$$k_1 t \left.\frac{\partial^2 T_2}{\partial x^2}\right|_{y=0} + k_2 \left.\frac{\partial T_2}{\partial y}\right|_{y=0} = 0 \tag{7c}$$

and

$$\left.\frac{\partial T_2}{\partial y}\right|_{y=\delta} = \frac{k}{k_2}(T_e - T_2(x, \delta)). \tag{7d}$$

Now, we define the dimensionless variables as

$$\theta = \frac{T_2 - T_e}{T_b - T_e}, \quad x^* = \frac{x}{L} \quad \text{and} \quad y^* = \frac{y}{\delta}. \tag{8}$$

After changing variables, the boundary value problem for $T_2(x, y)$ is

$$\frac{\partial^2 \theta}{\partial x^{*2}} + \frac{L^2}{\delta^2}\frac{\partial^2 \theta}{\partial y^{*2}} = 0 \quad \text{in } 0 < x^* < 1,\ 0 < y^* < 1 \tag{9}$$

with

$$\frac{\partial \theta}{\partial x^*} = 0 \quad \text{at } x^* = 1, \tag{10a}$$

$$\theta = 1 \quad \text{at } x^* = 0, \tag{10b}$$

$$\frac{k_1 t}{L^2}\frac{\partial^2 \theta}{\partial x^{*2}} + \frac{k_2}{\delta}\frac{\partial \theta}{\partial y^*} = 0 \quad \text{at } y^* = 0 \tag{10c}$$

and

$$\frac{\partial \theta}{\partial y^*} + \frac{h\delta}{k_2}\theta = 0 \quad \text{at } y^* = 1. \tag{10d}$$

From this point forward, the superscript "∗" will be dropped from the spatial coordinates for convenience, with x and y taken as dimensionless unless otherwise noted. Notice that the boundary condition (10b) conflicts with (10d) at $(x, y) = (0, 1)$. In order to remove the singularity, replace (10b) with the following at $x = 0$:

$$\theta = f(y) = \begin{cases} 1, & 0 \leqslant y < (1 - \varepsilon), \\ 1 - \dfrac{[y - (1 - \varepsilon)]^2}{[(2\varepsilon k_2/h\delta) + \varepsilon^2]}, & (1 - \varepsilon) \leqslant y \leqslant 1, \end{cases} \tag{10e}$$

where $0 < \varepsilon \ll 1$. We have $(\partial\theta/\partial y)_{(0,1)} + (h\delta/k_2)\theta(0, 1) = 0$ and Eq. (10e) → Eq. (10b) as $\varepsilon \to 0$. Moreover, $f(y)$ is twice differentiable on $0 \leqslant y \leqslant 1$. The boundary condition at $x = 0$ can be generalized to be any twice-differentiable function $f(y)$ on $0 \leqslant y \leqslant 1$. Because only the boundary condition given by Eq. (10e) is nonhomogeneous, separation of variables is pursued. That is, assume

$$\theta(x, y) = X(x)Y(y), \tag{11}$$

then $X(x)$ should satisfy

$$X'' - \left(\frac{L\lambda}{\delta}\right)^2 X = 0 \quad \text{in } 0 < x < 1 \tag{12}$$

with

$$X' = 0 \quad \text{at } x = 1 \tag{13}$$

and $Y(y)$ satisfies

$$Y'' + \lambda^2 Y = 0 \quad \text{in } 0 < y < 1 \tag{14}$$

with

$$Y' + (h\delta/k_2)Y = 0 \quad \text{at } y = 1 \tag{15}$$

and

$$\frac{k_1 t}{L^2} X'' Y + \frac{k_2}{\delta} X Y' = 0 \quad \text{at } y = 0. \tag{16}$$

Together with Eq. (12), the boundary condition of Eq. (16) becomes

$$Y' + M\lambda^2 Y = 0 \quad \text{at } y = 0 \tag{17a}$$

with

$$M = \frac{k_1 t}{k_2 \delta}. \tag{17b}$$

The second-order ordinary differential equations for $X(x)$ and $Y(y)$ are solved, and three of the four constants are determined using the boundary conditions of Eqs. (13), (15) and (17). The solution is

$$\theta(x,y) = \sum_{n=1}^{\infty} C_n Y(\lambda_n; y) \cosh\left(\frac{\lambda_n L}{\delta}(1-x)\right), \tag{18}$$

where the eigenfunctions $Y(\lambda_n; y)$ are

$$Y(\lambda_n; y) = \cos(\lambda_n y) - M\lambda_n \sin(\lambda_n y), \tag{19}$$

and the eigenvalues λ_n satisfy the following eigencondition:

$$\tan(\lambda_n) = \frac{k_1 k_2 t}{\delta(k_2^2 + h k_1 t)} \left[\left(\frac{h\delta^2}{k_1 t}\right)\frac{1}{\lambda_n} - \lambda_n\right], \quad n = 1, 2, 3, \ldots \tag{20}$$

The last boundary condition of Eq. (10e) gives

$$f(y) = \sum_{n=1}^{\infty} C_n [\cos(\lambda_n y) - M\lambda_n \sin(\lambda_n y)] \cosh\left(\frac{\lambda_n L}{\delta}\right). \tag{21}$$

According to the sense of orthogonality derived in Appendix A,

$$C_m = \frac{\int_0^1 f(y') Y(\lambda_m; y')\, dy' + M f(0) Y(\lambda_m; 0)}{\cosh\left(\frac{\lambda_m L}{\delta}\right)\left\{\int_0^1 [Y(\lambda_m; y')]^2 dy' + M[Y(\lambda_m; 0)]^2\right\}}, \tag{22}$$

then as $\varepsilon \to 0$

$$C_m = \frac{\int_0^1 Y(\lambda_m; y')\, dy' + M Y(\lambda_m; 0)}{\cosh\left(\frac{\lambda_m L}{\delta}\right)\left\{\int_0^1 [Y(\lambda_m; y')]^2 dy' + M[Y(\lambda_m; 0)]^2\right\}}, \tag{23}$$

thus,

$$C_m = \frac{2\left(\frac{\sin(\lambda_m)/\lambda_m}{\cosh(\lambda_m L/\delta)} + M\frac{\cos(\lambda_m)}{\cosh(\lambda_m L/\delta)}\right)}{1 + \frac{\sin(2\lambda_m)}{2\lambda_m} + M^2\lambda_m^2\left[1 - \frac{\sin(2\lambda_m)}{2\lambda_m}\right] + M[1 + \cos(2\lambda_m)]}. \tag{24}$$

Finally, the temperature distribution inside the two-dimensional slab—using dimensional variables—is

$$\frac{T_2(x,y) - T_e}{T_b - T_e} = \sum_{n=1}^{\infty} C_n \left[\cos\left(\frac{\lambda_n y}{\delta}\right) - M\lambda_n \sin\left(\frac{\lambda_n y}{\delta}\right)\right] \times \cosh\left(\frac{\lambda_n (L-x)}{\delta}\right), \tag{25}$$

where λ_n and C_n are given by Eqs. (20) and (24), respectively. An expression for the temperature along the one-dimensional fin is obtained by evaluating Eq. (25) at $y = 0$

$$\frac{T_1(x) - T_e}{T_b - T_e} = \sum_{n=1}^{\infty} C_n \cosh\left(\frac{\lambda_n (L-x)}{\delta}\right). \tag{26}$$

The fin heat transfer can be calculated by differentiating Eq. (26) and using Fourier's law at $x = 0$ to find the heat flowing from material 1; likewise, we differentiate Eq. (25), apply Fourier's law at $x = 0$, and integrate from $y = 0$ to $y = \delta$ to find the heat flowing from material 2. Dividing the fin heat transfer by the convective heat transfer that would occur if the frost surface temperature were equal to the base temperature gives the fin efficiency

$$\eta = \frac{1}{\delta h L} \sum_{n=1}^{\infty} C_n \lambda_n \sinh\left(\frac{\lambda_n L}{\delta}\right) \times \left\{k_1 t + \delta k_2 \left(\frac{\sin(\lambda_n)}{\lambda_n} + M(\cos(\lambda_n) - 1)\right)\right\}. \tag{27}$$

The calculation of the temperature profile and fin efficiency using the above solutions is much easier than the solution for two-dimensional heat flow both for the fin and its coating material, especially when calculating the eigenvalues, λ_n. According to Barker [7], the solution to that more complex case results in the following eigencondition

$$\frac{k_1}{k_2}\tan(\lambda_n^B) = \frac{1 - \lambda_n^B \frac{k_2}{ht}\tan\left(\lambda_n^B\left(\frac{\delta}{t} - 1\right)\right)}{\tan\left(\lambda_n^B\left(\frac{\delta}{t} - 1\right)\right) + \lambda_n^B \frac{k_2}{ht}}. \tag{28}$$

The behavior of Eq. (28) is complex, depending on δ/t, and this complexity makes it difficult to develop an initial guess for the distribution of the roots of Eq. (28); therefore, calculation of eigenvalues through a Newton–Raphson iteration—or by any other method—is difficult. In contrast, by designating the left-hand side of Eq. (20) as $f_1(\lambda)$ and the right-hand side as $f_2(\lambda)$ the eigen-

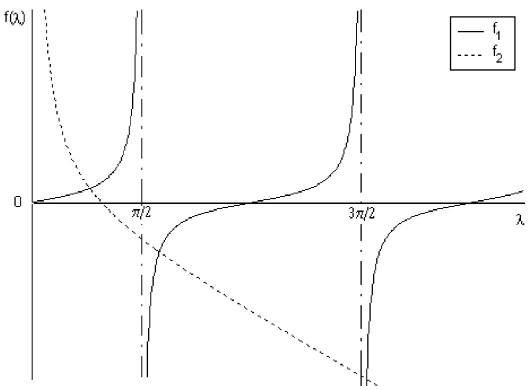

Fig. 2. The roots of Eq. (20) are shown as the intersection of the left-hand side (f_1) and the right-hand side (f_2) of the equation.

condition for the simplified problem always behaves as shown in Fig. 2. A clear expectation for the distribution of the roots is possible, and solution by Newton–Raphson is more likely to converge.

4.1. An example application to frost on an aluminum fin

The temperature distribution along a fin is calculated for frost on a metallic fin using the realistic parameters given in Table 2. The conditions of Table 2 make $Bi = 0.18$, $R = 8.2 \times 10^{-4}$, and $t \approx 0.1\delta \approx 0.01L$. According to Table 1, the temperature differences between $y = 0$ and $y = -t$ are less than 5% of $T_e - T_b$, and the fin can be assumed one-dimensional; thus, the analytical solution applies. Temperature results for this practical case were also obtained using a numerical solution to the fully two-dimensional case and a comparison of the numerical and analytical results of Eqs. (20), and (24)–(26) is given in Fig. 3. It is demonstrated in the figure that the full analytical solution matches the numerical solution very well. It is also evident that for $y = 0$, a one-term approximation is valid, but for $y = \delta$ an additional term is required to predict the temperature along the frost-air interface.

The fin efficiency is shown in Fig. 4, for two different values of the convection coefficient, $h = 50.4$ and 70.4 $\mathrm{W\,m^{-2}\,°C^{-1}}$, with a range of frost thicknesses, to expand

Table 2
Parameters of an example

T_e, °C	−2.8
T_b, °C	−5.0
L, mm	4.14
δ, mm	0.45
h, $\mathrm{W\,m^{-2}\,K^{-1}}$	70.4
k_1, $\mathrm{W\,m^{-1}\,K^{-1}}$	237
k_2, $\mathrm{W\,m^{-1}\,K^{-1}}$	0.175
t, mm	0.05

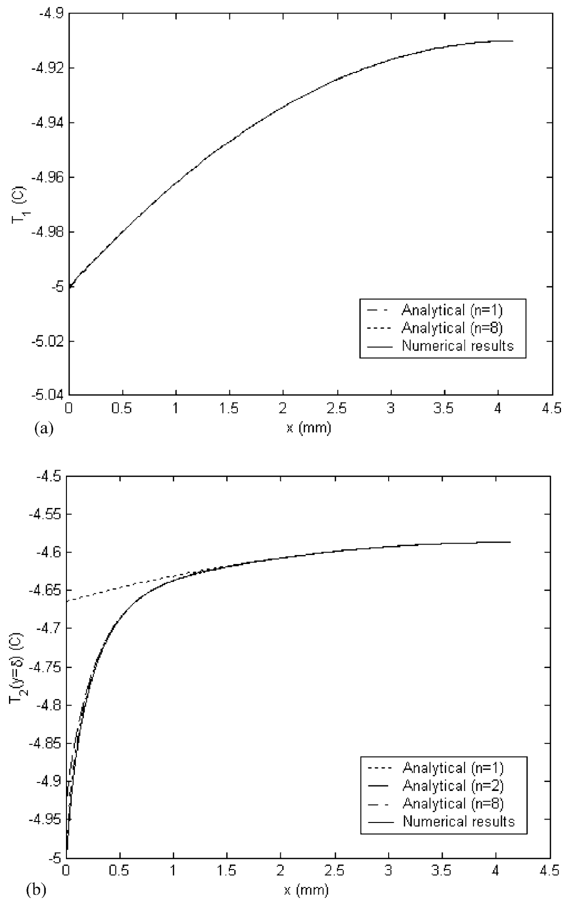

Fig. 3. A comparison of the numerical results to the analytical solution for the test conditions given in Table 2. Results are calculated using different numbers of terms in the series, for (a) $T_2(x,0) = T_1(x)$ and (b) $T_2(x,\delta)$.

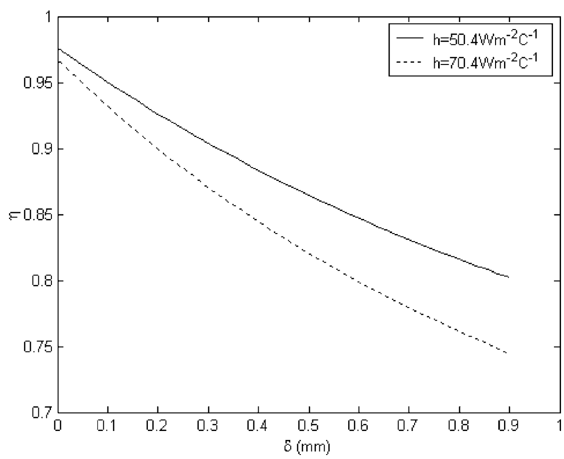

Fig. 4. Using the conditions of Table 2, for a range of frost thicknesses and two values of convective heat transfer coefficient, example fin efficiency results are provided.

the conditions of Table 2. These results were obtained with Eqs. (20), (24) and (27). The fin efficiency does not go to unity for a zero-thickness frost layer, because the metallic substrate is not a perfect conductor of heat. It should be noted that while the fin efficiency depends on t, δ, L, h, k_1 and k_2, it does not depend on the temperatures T_e and T_b.

4.2. One-term approximation

It is possible to consider a special case for which a one-term approximation to the series solution is sufficient by exploiting the behavior of the eigencondition. The first eigenvalue is always less than the positive root of $f_2(\lambda) = 0$. That is,

$$0 < \lambda_1 < \sqrt{\frac{h\delta^2}{k_1 t}}. \tag{29}$$

When

$$\sqrt{\frac{h\delta^2}{k_1 t}} \ll 1, \tag{30}$$

we have $\tan(\lambda_1) \approx \lambda_1$, and the first root of Eq. (20) can be approximated by

$$\lambda_1 \approx \delta\sqrt{\frac{h}{k_1 t + \delta(k_1 h t/k_2 + k_2)}}. \tag{31}$$

Furthermore, it can be shown from Eq. (24) that when

$$\frac{k_1 k_2 t}{\delta(k_2^2 + h k_1 t)} \gg 1, \tag{32}$$

we have

$$C_1 \approx \left[\cosh\left(\frac{\lambda_1 L}{\delta}\right)\right]^{-1} \tag{33}$$

with $C_n \approx 0$ for $n = 2, 3, \ldots$

The one-term approximation to the series solution in material 2 is then

$$\frac{T_2(x,y) - T_e}{T_b - T_e} \approx \left[1 - \frac{M\lambda_1^2 y}{\delta}\right] \frac{\cosh\left(\frac{\lambda_1(L-x)}{\delta}\right)}{\cosh\left(\frac{\lambda_1 L}{\delta}\right)}, \tag{34a}$$

and in material 1

$$\frac{T_1(x) - T_e}{T_b - T_e} \approx \frac{\cosh\left(\frac{\lambda_1(L-x)}{\delta}\right)}{\cosh\left(\frac{\lambda_1 L}{\delta}\right)}, \tag{34b}$$

where λ_1 is given by Eq. (31). Under the one-term approximation, the fin efficiency is

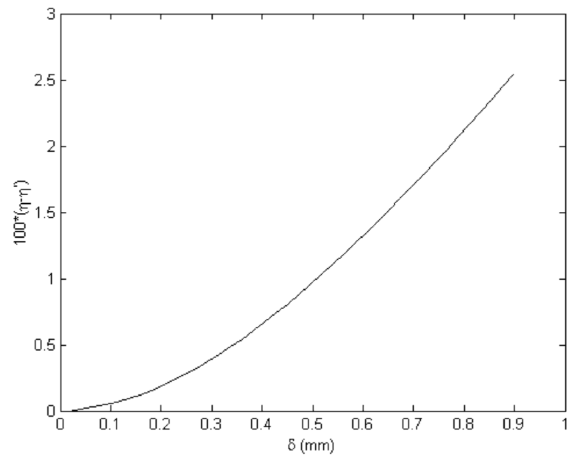

Fig. 5. The difference between fin efficiency accounting for conduction in the frost layer and fin efficiency assuming negligible conduction from the frost to the tube is shown. The plot is constructed using the conditions of Table 2.

$$\eta' = \frac{\lambda_1}{hL\delta} \frac{\sinh\left(\frac{\lambda_1 L}{\delta}\right)}{\cosh\left(\frac{\lambda_1 L}{\delta}\right)} (k_1 t + k_2\delta). \tag{35}$$

In Fig. 5, the difference between η and η' is shown as a function of the frost thickness for the conditions of Table 2. The one-term approximation under-predicts the fin efficiency by up to a few percent at the lowest fin efficiency. For fin efficiency larger than 80%, the series solution and its one-term approximation, Eqs. (27) and (35) respectively, differ by less than 1%. It is noteworthy that if the heat conducted into the base through the frost is neglected, then the η–η' is as high as 20% for some conditions used in Fig. 5. The error in neglecting conduction through the frost is pronounced for thick frost layers. Thus, Eq. (35) which accounts for conduction through both the frost and the metallic fin is preferred to an expression neglecting such effects.

5. Conclusions

The temperature distribution inside a two-dimensional composite fin is analyzed. A numerical parametric analysis shows that when $Bi > 0.05$ and $R < 0.1$, the problem can be approximated as a two-dimensional slab on a one-dimensional fin. Under this approximation, an exact solution is obtained by the separation of variables, exploiting orthogonality in the sense defined in Appendix A. In comparison to prior fully two-dimensional solutions, this new solution has the advantages of rapid convergence and relatively simple calculation. Moreover, conditions are developed under which a one-term approximation to the solution is sufficient, and it is found that frost on a metallic fin often falls into this range. The analytical solution presented in this paper, and the one-term approximation, have broad applicability in addition to their use for calculating fin efficiency for frost-coated fins.

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Appendix A

The eigenvalue problem

$$u'' + \gamma u = 0 \tag{A.1}$$

satisfying

$$u'(1) = c_1 u(1) \tag{A.2a}$$

and

$$u'(0) = \gamma c_2 u(0) \tag{A.2b}$$

is unusual, because the eigenvalue appears in the boundary condition. Following Friedman [10], who presented the solution to a similar problem, we consider the space of two-component vectors U whose first component is a real twice-differentiable function $u(x)$ and whose second component is a real number u_o. That is,

$$U = \begin{pmatrix} u(x) \\ u_o \end{pmatrix}. \tag{A.3}$$

Define the scalar product of two vectors U and V as

$$\langle U, V\rangle = \int_0^1 u(x)v(x)\,\mathrm{d}x - c_2 u_0 v_0, \tag{A.4}$$

and consider a subspace D of vectors U, such that

$$u'(1) = c_1 u(1) \tag{A.5a}$$

and

$$u(0) = u_o. \tag{A.5b}$$

If the linear operator, L, is defined such that

$$LU = \begin{pmatrix} -u''(x) \\ u'(0)/c_2 \end{pmatrix}, \tag{A.6}$$

then the eigenvalue problem is reduced to finding a vector U in D such that $LU = \gamma U$.

Moreover, with Eqs. (A.4) and (A.6) we have

$$\langle V, LU\rangle = -\int_0^1 v(x)u''(x)\,\mathrm{d}x - c_2 v_0 u'(0)/c_2, \tag{A.7a}$$

which gives

$$\langle V, LU\rangle = [-vu' + v'u]_0^1 - \int_0^1 uv''\,\mathrm{d}x - v(0)u'(0) \tag{A.7b}$$

or

$$\langle V, LU\rangle = -\int_0^1 uv''\,\mathrm{d}x - u_0 v'(0) = \langle LV, U\rangle. \tag{A.7c}$$

Eqs. (A.7) prove that L is self-adjoint, and therefore all the eigenvalues are real valued, and the eigenfunctions are orthogonal in the sense of Eq. (A.4), i.e.

$$\langle U_m, U_n\rangle \begin{cases} = 0, & m \neq n, \\ \neq 0, & m = n. \end{cases} \tag{A.8}$$

Any arbitrary function can be expressed as a series of the eigenfunctions, e.g. let

$$F = \begin{pmatrix} f(x) \\ f(0) \end{pmatrix} \quad \text{and} \quad U_n = \begin{pmatrix} u_n(x) \\ u_n(0) \end{pmatrix} \tag{A.9}$$

be vectors in D, where $u_n(x)$ is the solution of the original differential equation, corresponding to eigenvalue γ_n. Then we have the expansion

$$F = \sum \alpha_n U_n \tag{A.10}$$

or

$$f(x) = \sum_1^\infty \alpha_n u_n(x) \quad \text{and} \quad f(0) = \sum_0^\infty \alpha_n u_n(0), \tag{A.11}$$

where the coefficients α_n's are determined by

$$\alpha_n = \frac{\langle F, U_n\rangle}{\langle U_n, U_n\rangle} = \frac{\int_0^1 f(x)u_n(x)\,\mathrm{d}x - c_2 f(0)u_n(0)}{\int_0^1 [u_n(x)]^2\,\mathrm{d}x - c_2[u_n(0)]^2}. \tag{A.12}$$

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